NON-CROSSING PARTITIONS, NON-NESTING PARTITIONS AND COXETER SORTABLE ELEMENTS IN TYPES A AND B

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Abstract. First, we investigate a generalization of the area statistic on Dyck paths for all crystallographic reflection groups. In particular, we explore Dyck paths of type B together with an area statistic and a major index. Then, we construct bijections between non-nesting and reverse non-crossing partitions for types A and B. These bijections simultaneously send the area statistic and the major index on non-nesting partitions to the length function and to the sum of the major index and the inverse major index on reverse non-crossing partitions. Finally, we construct bijections between non-nesting partitions and Coxeter sortable elements for types A and B having exactly the same properties.

Contents

Date: August 20, 2008.
2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05A18, 05E15, 20F55.
Key words and phrases. non-crossing partitions, non-nesting partitions, Coxeter sortable elements, pattern-avoiding permutations, Dyck paths, permutation statistics.
Research supported by the Austrian Science Foundation FWF, grant P17563-N13 "Macdonald polynomials and q-hypergeometric Series" and grant S9600 "Analytic Combinatorics and Probabilistic Number Theory".

Introduction

The lattice of non-crossing set partitions can be seen as a well-behaved sublattice of the intersection lattice for the hyperplane arrangement of type A, see e.g. [16, 17, 30, 31]. In [29], V. Reiner generalized this construction to all classical reflection groups and T. Brady and C. Watt considered non-crossing partitions for any (finite) real reflection group W as certain intervals in the poset $(W, \leq_T)$ where $\leq_T$ is the total order on W, see [11, 12].

Recently, N. Reading introduced the notion of Coxeter sortable elements in a reflection group W [27]. Furthermore, he constructed bijections to non-crossing partitions on the one hand and to clusters in the cluster complex on the other hand using Coxeter sortable elements. This simplicial complex was constructed by S. Fomin and A. Zelevinsky in the context of cluster algebras which arose within the last years in more and more contexts in various fields of mathematics, see [18, 19, 20, 21].

Non-nesting partitions were introduced simultaneously for all crystallographic reflection groups by A. Postnikov, see [29, Remark 2]. Furthermore, he connected them to the Shi arrangement which has many interesting properties and which generalizes the Coxeter arrangement.

The work on non-nesting partitions on the one hand and on non-crossing partitions respectively on Coxeter sortable elements on the other hand suggests that they are not only counted by the same numbers, namely the Catalan numbers associated to a reflection group, but are deeply connected, at least in a numerical sense. These connections were nicely described by D. Armstrong in [4, Chapter 5.1.3]. Except for type A, no bijections between these objects were known so far and these connections are still mysterious and not well understood.

In this paper, we investigate bijections between non-nesting partitions, reverse non-crossing partitions and Coxeter sortable elements in types A and B. These bijections are interesting in the sense that they nicely translate natural statistics on non-nesting partitions, namely the area statistic and the major index, to natural statistics on non-crossing partitions and on Coxeter sortable elements respectively.

In Section 1, we introduce the objects we need and present several of their properties. In particular, we define q-Catalan numbers for crystallographic reflection groups (see Definition 1.10) as well as Dyck paths of type B together with an area statistic and a major index (see Definitions 1.12 and 1.18). Furhermore, we investigate their recurrence relation and their generating function (see Theorem 1.15 and Corollary 1.17) and their connection to known q-Catalan numbers in this context (see Proposition 1.21).

In Section 2, we construct bijections between non-nesting and reverse non-crossing partitions in types A and B. They simultaneously send the area statistic and the major index on non-nesting partitions to the length function and to the sum of the major index and the inverse major index on reverse non-crossing partitions, see Theorem 2.1.

In Section 3, we construct analogous bijections between non-nesting partitions and Coxeter sortable elements in types A and B having exactly the same properties, see Theorem 3.1.

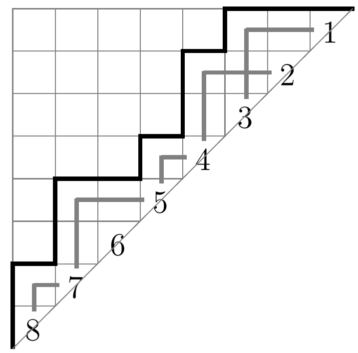


Figure 1. A Dyck path of length 8 and the associated non-nesting set partition $\{\{1,3\},\{2,4,5,7,8\},\{6\}\}$.

Acknowledgements

The author would like to thank Drew Armstrong, Riccardo Biagioli, Christian Krattenthaler, Nathan Reading and Martin Rubey for useful conversations.

The results in this paper are part of a PhD thesis written at the University of Vienna, Austria [35]. The results in Section 1.2 were part of a presentation at the FPSAC 2008 conference in Viña del Mar, Chile, see [34].

1. Background and definitions

One of the most famous and most studied integer sequences in combinatorics is the sequence of Catalan numbers. These are defined by

$$\mathrm{Cat}_n := \frac{1}{n+1}\binom{2n}{n}.$$

In his book "Enumerative Combinatorics Vol. 2" [32], R.P. Stanley lists more than 66 combinatorial interpretations of these numbers. We start with 4 basic occurrences of the Catalan numbers which can be found in [32, 6.19 (h), (pp), (uu), (ff)]:

Example 1.1 (Dyck paths)**.** A Dyck paths of length n is defined to be a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) consisting of n north steps of the form $(0,1)$ and n east steps of the form $(1,0)$ with the property that the path never goes below the line $x=y$. Denote the set of all Dyck paths of length n by $\mathcal{D}_n$. As we need them later, we give two alternative descriptions of Dyck paths:

- a Dyck path can be encoded by a word D consisting of n N's and n E's where any prefix of D does not contain more E's than N's and
- a Dyck path can be identified with a partition, i.e., a weakly decreasing sequence of non-negative integers, fitting inside the partition $(n-1,\dots,2,1,0)$.

In Figure 1, a Dyck path of length 8 is shown. It is encoded by the word

$$NNENNEENENNENEEE$$

and the associated partition is

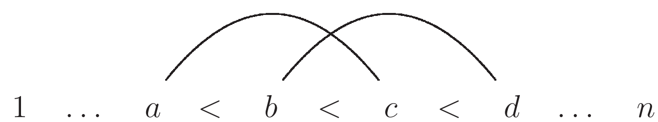
$$\lambda = (5,4,4,3,1,1) \subseteq (7,6,5,4,3,2,1,0).$$

The number of Dyck paths of length n equals the n-th Catalan number, $|\mathcal{D}_n| = \mathrm{Cat}_n$.

Example 1.2 (Non-crossing set partitions)**.** Let $[n] := \{1, 2, 3, \ldots, n\}$ be the set of the first n integers. A set partition of $[n]$ is a partition of $[n]$ into non-empty pairwise disjoint subsets, called blocks, $B_1, \ldots, B_k$ with $\bigcup B_i = [n]$. A set partition $\{B_1, \ldots, B_k\}$ is non-crossing if

$$a < b < c < d \text{ with } a, c \in B_i \text{ and } b, d \in B_j \text{ implies } B_i = B_j.$$

The set of all non-crossing set partitions of $[n]$ is denoted by $NC(n)$. Often, a set partition $\mathcal{B}$ is visualized by drawing the numbers 1 to n in a row and then drawing arcs on top of two adjacent elements in a block of $\mathcal{B}$. Then the condition for a set partition to have two crossing blocks is visualized as follows:

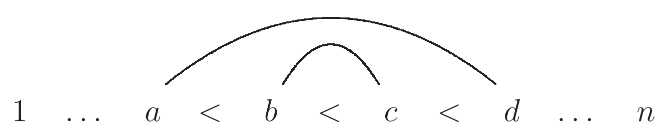


There exists a nice and simple bijection between non-crossing set partitions and Dyck paths via non-nesting set partitions:

Example 1.3 (Non-nesting set partitions)**.** A set partition $\{B_1, \ldots, B_k\}$ is non-nesting if

$$a < b < c < d \text{ with } a, d \in B_i \text{ and } b, c \in B_j \text{ implies } B_i = B_j.$$

The set of all non-nesting set partitions of $[n]$ is denoted by $NN(n)$. The condition for a set partition to have two nesting blocks is visualized as follows:



The intuitive map that locally converts each nesting into a crossing defines a bijection between non-crossing and non-nesting set partitions and the map indicated in Figure 1 is a bijection between Dyck paths and non-nesting set partitions (see e.g. [4, Section 5.1.2]).

Example 1.4 (3-pattern-avoiding permutations)**.** For a permutation $\sigma \in \mathcal{S}_n$, the one-line notation of σ is the presentation of σ as the list $[\sigma_1, \ldots, \sigma_n]$ where $\sigma_i := \sigma(i)$. A subword of σ is a subsequence $[\sigma_{i_1}, \ldots, \sigma_{i_k}]$ with $i_1 < \ldots < i_k$ of σ. For $\tau \in \mathcal{S}_k$, σ is called τ-avoiding if σ does not contain a subword of length k having the same relative order as τ. By $\mathcal{S}_n(\tau)$, we denote the set of of τ-avoiding permutations in $\mathcal{S}_n$. In [24], D.E. Knuth proved for any $\tau \in \mathcal{S}_3$ that the number of τ-avoiding permutations in $\mathcal{S}_n$ is equal to Cat_n.

The objects defined in Examples 1.1–1.4 can be seen as the type A instances of more general constructions called *non-nesting partitions*, *non-crossing partitions* and *Coxeter sortable elements* which can be attached to certain classes of reflection groups

A_{n-1}	B_n	D_n	$I_2(k)$	H_3	H_4	F_4	E_6	E_7	E_8
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n}{n}$	$\binom{2n}{n}-\binom{2n-2}{n-1}$	$k+2$	32	280	105	833	4160	25080

FIGURE 2. Cat(W) for all irreducible real reflection groups.

and which are all counted by a suitable generalization of the Catalan numbers. In the remaining part of this section, we will define these objects and we will explore some of their properties.

1.1. **Catalan numbers for (finite) reflection groups.** The Catalan numbers are naturally associated to the symmetric group $\mathcal{S}_n$ which is the reflection group of type A_{n-1}. For any (finite) real reflection group W, define the sequence of Catalan number associated to W as

$$\mathrm{Cat}(W) := \prod_{i=1}^{l} \frac{d_i + h}{d_i},$$

where l is the *rank* of W, $d_1 \leq \ldots \leq d_l$ are its *degrees* and h is its *Coxeter number*. For an introduction to real reflection groups and for definitions see [23].

In Figure 2, the Catalan numbers $\mathrm{Cat}(W)$ are shown for all irreducible real reflection groups.

In type A, a first q-extension of Cat we want to draw attention to is defined by

$$\text{q-Cat}_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

where $[k]_q := 1 + q + \ldots + q^{k-1}$ is the usual q-extension of an integer k, $[k]_q! := [1]_q[2]_q \cdots [k]_q$ is the q-factorial of k and $\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]_q := [k]_q!/[l]_q![k-l]_q!$ is the q-binomial coefficient.

P.A. MacMahon showed that $\text{q-Cat}_n(q)$ is equal to the generating function for the major index on Dyck paths, see e.g. [26]: recall from Example 1.1, that a Dyck path can be encoded by a word in the alphabet $\{N, E\}$ and set $N < E$. For a Dyck path D of length n, the major index of D is defined by

$$\mathrm{maj}(D) := \sum_{i \in \mathrm{Des}(D)} (2n - i),$$

where for any word $\omega = \omega_1 \cdots \omega_k$ in a totally ordered alphabet, $\mathrm{Des}(\omega)$ is defined by $\mathrm{Des}(\omega) := \{i : \omega_i > \omega_{i+1}\}$, compare Section 1.3. Then

$$\sum_{D \in \mathcal{D}_n} q^{\mathrm{maj}(D)} = \text{q-Cat}_n(q).$$

Remark. When only dealing with Dyck paths, the major index of a given path D is usually defined as $\sum_{i \in \mathrm{Des}(D)} i$. The involution $\mathbf{c}$ on Dyck paths sending a path to the path obtained by reversing the associated word in $\{N, E\}$ and then interchanging the N-th and E-th gives the same generating function. As $\mathbf{c}$ can equivalently described by the involution which conjugates the associated partition, we call $\mathbf{c}(D)$ the conjugate of

D. In Section 1.2 we will introduce Dyck paths of type B_n and we will see that in this context, the given definition is more convenient.

Using the involution $\mathbf{c}$ on Dyck paths, we derive the following proposition:

Proposition 1.5. *The sequence of coefficients of the q-Catalan numbers* $\text{q-Cat}_n(q)$ *is symmetric, i.e.,*

$$\text{q-Cat}_n(q) = q^{n(n-1)}\,\text{q-Cat}_n(q^{-1}).$$

At the workshop "Braid groups, clusters and free probability" [3], Athanasiadis suggested to generalize this q-extension of Cat_n as

$$\text{q-Cat}(W;q) := \prod_{i=1}^{l} \frac{[d_i+h]_q}{[d_i]_q}.$$

In type A, this product reduces to $\text{q-Cat}_n(q)$,

$$\text{q-Cat}(A_{n-1};q) = \text{q-Cat}_n(q).$$

Remark. For general W, these q-Catalan numbers seem to have first appeared in a paper by Y. Berest, P. Etingof and V. Ginzburg [8] where it is obtained as a certain Hilbert series. Their work implies that this extension is in fact a polynomial with non-negative integer coefficients.

1.2. **Non-nesting partitions and Dyck paths.** One can define non-nesting partitions for any crystallographic reflection group W in terms of its root poset: let $\Delta \subseteq \Phi^+$ be a simple respectively positive system of roots for W. Define a covering relation $\prec$ on Φ^+ by

$$\alpha \prec \beta :\Leftrightarrow \beta - \alpha \in \Delta.$$

This covering relation turns Φ^+ into a poset, the root poset associated to W.

Before defining non-nesting partitions, we observe that a Dyck path D can be identified with the collection of cells $b_{i,j}$ which lie strictly below D and strictly above the line $x = y$, where for $0 \leq i < j < n$ the cell $b_{i,j} \subseteq \mathbb{R}^2$ is given by

$$b_{i,j} := \big\{(x,y) \in \mathbb{R}^2 : i < x < i+1, j < y < j+1\big\}.$$

Example 1.6. The Dyck path shown in Figure 1 on page 3 can be identified with the collection of cells given by

$$\big\{b_{01}, b_{12}, b_{23}, b_{34}, b_{45}, b_{56}, b_{67}, b_{13}, b_{46}, b_{57}\big\}.$$

To see how this construction can be interpreted in terms of the root poset Φ^+ of type A_{n-1}, observe that for $1 \leq i < j \leq n$, the map sending the positive root $\epsilon_j - \epsilon_i \in \Phi^+$ to the cell $b_{n-j,n-i}$ defines a bijection between order ideals in Φ^+ and Dyck paths of length n, where an order ideal is a subset $I \subseteq \Phi^+$ with the property that $\beta \leq \alpha \in I$ implies $\beta \in I$. It is denoted by $I \trianglelefteq \Phi^+$.

Example (continued) 1.7. The order ideal associated to the Dyck path shown in Figure 1 on page 3 is given by

$$\big\{\epsilon_8 - \epsilon_7, \epsilon_7 - \epsilon_6, \epsilon_6 - \epsilon_5, \epsilon_5 - \epsilon_4, \epsilon_4 - \epsilon_3, \epsilon_3 - \epsilon_2, \epsilon_2 - \epsilon_1, \epsilon_7 - \epsilon_5, \epsilon_4 - \epsilon_2, \epsilon_3 - \epsilon_1\big\}.$$

We use this interpretation of Dyck paths as the motivation to define non-nesting partitions as follows:

Definition 1.8. Let W be a crystallographic reflection group with root poset Φ^+. The non-nesting partition lattice $NN(W) := \{I \trianglelefteq \Phi^+\}$ is the collection of all order ideals in Φ^+ ordered by inclusion.

Remark. Originally, $NN(W)$ was defined by A. Postnikov as *antichains* in the root poset, i.e., subsets of pairwise non-comparable elements, see [29, Remark 2]. Sending an order ideal to its maximal elements gives a natural bijection between order ideals and antichains.

The following theorem is due to Postnikov, see [29, Remark 2]:

Theorem 1.9 (Postnikov)**.** *Let W be a crystallographic reflection group. Then*

$$|NN(W)| = \mathrm{Cat}(W).$$

We now want to introduce a q-extension of $\mathrm{Cat}(W)$ in terms of non-nesting partitions that generalizes the well-understood case of $W = A_{n-1}$. Later, we will discuss the situation in type B in more detail.

Definition 1.10. Let W be a crystallographic reflection group. Define

$$\mathrm{Cat}(W;q) := \sum_{I \in NN(W)} q^{|I|}.$$

Example 1.11. The non-nesting partition in Example 1.7 contributes q^{10} to $\mathrm{Cat}(A_7;q)$.

Remark. Non-nesting partitions are closely connected to a generalization of the Coxeter arrangement called Shi arrangement, see [5]. The q-extension defined above can also be nicely described in terms of this arrangement.

For $W = A_{n-1}$, the discussion at the beginning of this subsection implies that the number of elements in a given order ideal $I \trianglelefteq \Phi^+$ is equal to the number of cells b_{ij} which lie below the Dyck path associated to I. This statistic is called area statistic and was studied by J. Fürlinger and J. Hofbauer in [22]. It is one of the most studied statistics on Dyck paths. They derived the following recurrence relation:

$$\mathrm{Cat}_{n+1}(q) = \sum_{k=0}^{n} q^k \,\mathrm{Cat}_k(q)\,\mathrm{Cat}_{n-k}(q), \qquad \mathrm{Cat}_0(q) = 1,$$

where $\mathrm{Cat}_n(q) := \mathrm{Cat}(A_{n-1};q) = \sum_{D \in \mathcal{D}_n} q^{\mathrm{area}(D)}$.

Following the idea of identifying order ideals in the root poset with lattice paths in $\mathbb{Z}^2$ such that the number of elements in an order ideal matches with the number of cells confined by the path together with other restrictions, we define Dyck paths of type B together with an area statistic and we establish an analogous recurrence as for Dyck paths of type A. Furthermore, we will describe why we were not able to construct Dyck paths of type D.

Definition 1.12. A Dyck path of type B_n is a lattice paths of $2n$ steps, either north or east, that starts at $(0,0)$ and stays above the diagonal $x = y$. For such a path

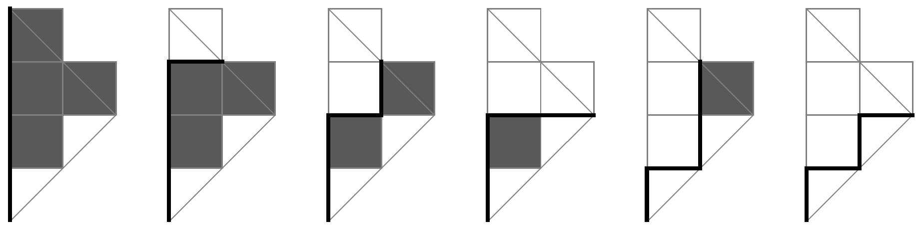


FIGURE 3. All type B Dyck paths of length 2.

D, we define $\operatorname{area}(D)$ to be the number of cells b_{ij} that lie below D, but now with the additional property that $1 \leq i < j \leq 2n+1-i$. Furthermore, define q-Catalan numbers of type B_n by

$$\operatorname{Cat}_{B_n}(q) := \sum q^{\operatorname{area}(D)},$$

where the sum ranges over all Dyck paths of type B_n.

Example 1.13. In Figure 3, we list all Dyck paths of type B_2. The cells which contribute to the area are shaded. Therefore, we have

$$\operatorname{Cat}_{B_2}(q) = q^4 + q^3 + q^2 + 2q + 1.$$

Definition 1.12 implies the following proposition:

Proposition 1.14. *The q-Catalan numbers* $\operatorname{Cat}(W;q)$ *reduces in type B to* $\operatorname{Cat}_{B_n}(q)$,

$$\operatorname{Cat}(B_n;q) = \operatorname{Cat}_{B_n}(q).$$

The q-Catalan numbers of type B satisfy the following recurrence involving q-Catalan numbers of type A:

Theorem 1.15.

$$\operatorname{Cat}_{B_n}(q) = \operatorname{Cat}_n(q) + \sum_{k=0}^{n-1} q^{2k+1} \operatorname{Cat}_{B_k}(q) \operatorname{Cat}_{n-k}(q), \qquad \operatorname{Cat}_{B_0}(q) = 1.$$

Proof. Let D be a Dyck path of type B_n. Then either D has as many east as north steps, which means it is equal to a type A Dyck path of length n, or there exists a last point $(k, k+1)$ where the path touches the diagonal $x+1=y$ and stays strictly above afterwards. Now, we have an initial type A Dyck path of length $k+1$, except that the last step is a north step instead of an east step, see Figure 4 for an example. After this north step, a Dyck path of type B_{n-k-1} starts. This gives

$$\begin{aligned} \operatorname{Cat}_{B_n}(q) &= \operatorname{Cat}_n(q) + \sum_{k=0}^{n-1} q \operatorname{Cat}_{k+1}(q) q^{2(n-k-1)} \operatorname{Cat}_{B_{n-k-1}}(q) \\ &= \operatorname{Cat}_n(q) + \sum_{k=0}^{n-1} q^{2k+1} \operatorname{Cat}_{B_k}(q) \operatorname{Cat}_{n-k}(q) \end{aligned}$$

□

Example 1.16. Figure 4 shows a Dyck path of type B_6. It starts with a type A like Dyck path of length 3, ending with the north step between the two dots, followed by a Dyck path of type B_3, which starts after the second dot.

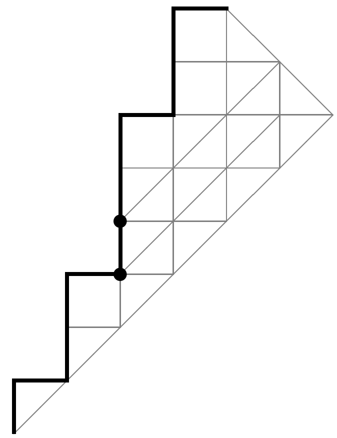


FIGURE 4. A Dyck path of type B_6.

Corollary 1.17. $\mathrm{Cat}_{B_n}(q)$ *satisfies the following generating function identity:*

$$\sum_{n\geq 0} \frac{x^n q^{-n(n-1)}(1-qx)}{(-x;q^{-1})_{2n+1}} \mathrm{Cat}_{B_n}(q) = 1.$$

Proof. The recurrence in Theorem 1.15 can be written as

$$(1+q^{2n+1})\,\mathrm{Cat}_{B_n}(q) - \sum_{k=0}^{n} q^{2k+1}\,\mathrm{Cat}_{B_k}(q)\,\mathrm{Cat}_{n-k}(q) = \mathrm{Cat}_n(q).$$

Multiplying both sides of the equality by $\frac{x^n q^{-n(n-1)}}{(-x;q^{-1})_{2n+1}}$ and summing over all n gives the proposed generating function identity. □

Remark. The root poset of type D_n is *not* planar for $n \geq 4$. Therefore, the construction of Dyck paths of type D in the above sense fails.

In the remaining part of this section, we want to exhibit a major index on Dyck paths of type B_n for which, in analogy to type A, the generating function is equal to q-Cat$(B_n;q)$.

It is well-known that for the major index on lattice paths consisting on n north and n east steps without any further restrictions we have

$$\sum q^{\mathrm{maj}(L)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q ,$$

where the major index is defined with respect to the order $E < N$ by

$$\mathrm{maj}(L) := \sum_{i\in Des(D)} (2n-i),$$

see e.g. [2] and [26].

Definition 1.18. Define the major index of a Dyck path D of type B_n as

$$\mathrm{maj}(D) := 2\cdot\big(\,\mathrm{neg}(D) + \sum_{i\in \mathrm{Des}(D)} (2n-i)\big),$$

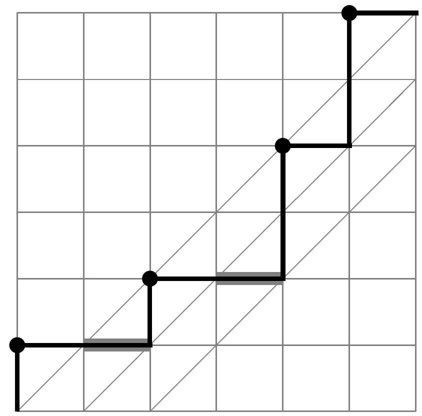


FIGURE 5. The lattice path from $(0,0)$ to $(6,6)$ which is mapped to to Dyck path of type B_6 shown in Figure 4.

where $\mathrm{neg}(D)$ is the number of east steps in D and $\mathrm{Des}(D)$ is, as for Dyck paths of type A, the descent set with respect to the order $N < E$.

Example 1.19. The Dyck path D of type B_6 shown in Figure 4 is encoded by the word

$$NE\ NNE\ NNNE\ NNE.$$

This gives $\mathrm{neg}(D) = 4$, $\mathrm{Des}(D) = \{2, 5, 9\}$ and therefore,

$$\mathrm{maj}(D) = 2(4 + (12 - 2) + (12 - 5) + (12 - 9)) = 2(4 + 10 + 7 + 3) = 48.$$

Note. Equivalently, we could have defined the major index on Dyck paths of type B by $\mathrm{maj}(D) := 2\,\mathrm{maj}(w)$, where w is the reverse of the Dyck word of D. In the previous example, we would have $w = E\ NNE\ NNNE\ NNE\ N$ and therefore, $\mathrm{Des}(w) = \{1, 4, 8, 11\}$, $2\,\mathrm{maj}(w) = 2(1 + 4 + 8 + 11) = 48 = \mathrm{maj}(D)$.

Definition 1.20.

$$\text{q-Cat}_{B_n}(q) := \sum q^{\mathrm{maj}(D)},$$

where the sum ranges over all Dyck paths of type B_n.

Proposition 1.21. *The generating function for the major index on Dyck paths of type B_n is equal to* $\text{q-Cat}(B_n; q)$,

$$\text{q-Cat}_{B_n}(q) = \text{q-Cat}(B_n; q).$$

Proof. Define a bijection between lattice paths form $(0,0)$ to (n,n) to Dyck paths of type B_n by replacing the first east step from level i to level $i-1$ by a north step for all $i < 0$ for which such an east step exists. For example, the lattice path shown in Figure 5 is mapped to the Dyck path shown in Figure 4. This transformation does not affect the major index with respect to the ordering $E < N$. For a lattice path L and

its image D, we then have $\operatorname{maj}(D) = 2\operatorname{maj}(L)$ and therefore

$$\text{q-Cat}_{B_n}(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}.$$

This can easily be seen to be equal to $\text{q-Cat}(W;q)$. $\square$

Example (continued) 1.22. The major index of the lattice path L shown in Figure 5 is given by

$$\operatorname{maj}(L) = (12-1) + (12-4) + (12-8) + (12-11) = 11 + 8 + 4 + 1 = 24.$$

As we have seen in Example 1.19, this gives $\operatorname{maj}(D) = 2\operatorname{maj}(L)$.

The following proposition is the analogue of Proposition 1.5:

Proposition 1.23. *The sequence of coefficients of the q-Catalan numbers* $\text{q-Cat}_{B_n}(q)$ *is symmetric, i.e.,*

$$\text{q-Cat}_{B_n}(q) = q^{2n^2}\,\text{q-Cat}_{B_n}(q^{-1}).$$

Proof. We have seen that $\text{q-Cat}_{B_n}(q) = \sum q^{2\operatorname{maj}(D)}$, where the sum ranges over all lattice paths from $(0,0)$ to (n,n) without any restrictions. The statement follows by applying the same involution as in type A to such a lattice path. $\square$

1.3. **Some statistics on classical reflection groups.** In this subsection, we want to present some statistics on the symmetric group, on the group of signed permutations an on the group of even signed permutations which are the reflection groups of types A, B and D respectively.

Define the inversion number on finite words in a totally ordered alphabet by

$$\operatorname{inv}(w) := \left|\left\{i < j : w_i > w_j\right\}\right|$$

for any word $w = w_1 w_2 \cdots w_k$. The inversion number can be used to compute the usual length function on permutations, signed permutations and even-signed permutations. We have

$$\begin{aligned} A_{n-1} : \operatorname{l}_S(\sigma) &= \operatorname{inv}([\sigma_1,\ldots,\sigma_n]), \\ B_n : \operatorname{l}_S(\sigma) &= \operatorname{inv}([\sigma_1,\ldots,\sigma_n]) - \sum_{i\in\operatorname{Neg}(\sigma)} \sigma_i, \\ D_n : \operatorname{l}_S(\sigma) &= \operatorname{inv}([\sigma_1,\ldots,\sigma_n]) - \sum_{i\in\operatorname{Neg}(\sigma)} \sigma_i - \operatorname{neg}(\sigma), \end{aligned}$$

where $\operatorname{Neg}(\sigma) := \{i \in [n] : \sigma_i < 0\}$ and $\operatorname{neg}(\sigma) := |\operatorname{Neg}(\sigma)|$. Type A is classical and was proved by MacMahon, see [26], whereas types B and D were proved in [14] by Brenti.

Next, we define the major index on words in a totally ordered alphabet. Let $w = w_1 w_2 \cdots w_k$ be a finite word, its descent set $\operatorname{Des}(w)$ is the set of all integers i such that $w_i > w_{i+1}$, $\operatorname{des}(w)$ denotes the cardinality of $\operatorname{Des}(w)$, and the major index of w is defined as

$$\operatorname{maj}(w) := \sum_{i\in\operatorname{Des}(w)} i.$$

Remark. The term "major index" was first used by D. Foata to indicate its origin, as MacMahon was a major in the British Army in the early 20th century, as well as the idea of counting positions of certain "major" elements. MacMahon himself used the term *greater index*.

Definition 1.24. Let σ be an element in the reflection group of a classical type with one-line notation $[\sigma_1, \ldots, \sigma_n]$. The major index of σ is then defined as

$$\begin{aligned} A_{n-1} : \mathrm{maj}(\sigma) &:= \mathrm{maj}([\sigma_1, \ldots, \sigma_n]), \\ B_n : \mathrm{maj}(\sigma) &:= 2\,\mathrm{maj}([\sigma_1, \ldots, \sigma_n]) + \mathrm{neg}(\sigma), \\ D_n : \mathrm{maj}(\sigma) &:= \mathrm{maj}([\sigma_1, \ldots, \sigma_n]) - \sum_{i \in \mathrm{Neg}(\sigma)} \sigma_i - \mathrm{neg}(\sigma). \end{aligned}$$

Remark. For permutations, this definition appeared first in [26]. For signed permutations, the major index was introduced by R.M. Adin and Y. Roichman in [1]. For even-signed permutations, the major index was introduced in a slightly different way by R. Biagioli in [9], the definition we use was introduced by Biagioli and F. Caselli in [10]. In [1] and [9], the major index for types B and D was called *f-major index* and in [10], the major index for type D was called *d-major index*. This was done to distinguish between different "major-like" statistics they were studying.

In [1, 10, 26], the following result was proved for classical reflection groups:

Theorem 1.25. *Let W be one of the reflection groups A_n, B_n or D_n. The major index on W is equally distributed with the length function l_S,*

$$\sum_{\sigma \in W} q^{\mathrm{maj}(\sigma)} = \sum_{\sigma \in W} q^{\mathrm{l}_S(\sigma)}.$$

For later convenience, we define also

$$\mathrm{iDes}(\sigma) := \mathrm{Des}(\sigma^{-1}), \mathrm{ides}(\sigma) := \mathrm{des}(\sigma^{-1}) \text{ and } \mathrm{imaj}(\sigma) := \mathrm{maj}(\sigma^{-1}),$$

the later is called inverse major index.

1.4. **Non-crossing partitions.**

Definition 1.26. Let W be a real reflection group with root system Φ and let

$$T := \{s_\alpha : \alpha \in \Phi^+\} \subseteq W = W(\Phi)$$

be the set of reflections in W. The absolute length of a given $\omega \in W$, denoted by $\mathrm{l}_T(\omega)$, is the smallest k for which ω can be written as a product of k reflections.

As S, the set of simple reflections, is contained in T, it immediately follows

$$\mathrm{l}_T(\omega) \leq \mathrm{l}_S(\omega) \text{ for all } \omega \in W,$$

where l_S is the ordinary length function on W with respect to S.

Using l_T, define a partial order on W by

$$\omega \leq_T \tau :\Leftrightarrow \mathrm{l}_T(\tau) = \mathrm{l}_T(\omega) + \mathrm{l}_T(\omega^{-1}\tau), \tag{1}$$

and denote the resulting poset by $\mathsf{Abs}(W)$. It is a graded poset with rank function l_T and unique minimal element $1 \in W$ but $\mathsf{Abs}(W)$ is *not* a lattice as in general it does

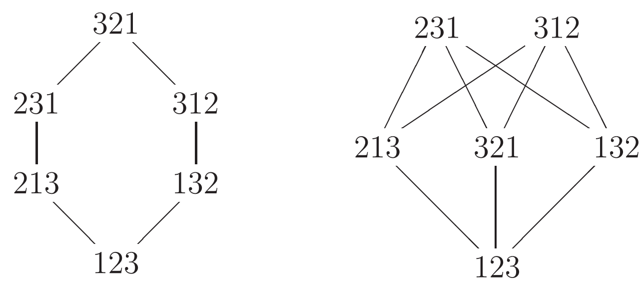


FIGURE 6. $\mathsf{Weak}(A_2)$ and $\mathsf{Abs}(A_2)$.

not have a unique maximal element. Figure 6 shows the Hasse diagrams of $\mathsf{Weak}(A_2)$ and $\mathsf{Abs}(A_2)$, where $\mathsf{Weak}(W)$ denotes the poset obtained from (1) by replacing l_T by l_S.

Note. By construction, the Coxeter elements of W have maximal absolute order or, equivalently, all Coxeter elements are among the top elements in $\mathsf{Abs}(W)$.

For the classical reflection groups, the absolute length can be combinatorially computed using the cycle notation: a cycle $(i_1, i_2, \ldots, i_k)$ is a list of integers in $\{\pm 1, \ldots, \pm n\}$ with distinct absolute values except for i_k which can possibly be equal to $-i_1$. We can regard a cycle as a signed permutation as follows: if $i_k \neq -i_1$ we obtain a signed permutation consisting of the cycles $i_1 \mapsto i_2 \mapsto \ldots \mapsto i_k \mapsto i_1$ and its negative analogue. If $i_k = -i_1$ we obtain a signed permutation constisting of the cycle $i_1 \mapsto i_2 \mapsto \ldots \mapsto i_k \mapsto -i_1 \mapsto -i_2 \mapsto \ldots \mapsto -i_k \mapsto i_1$. In both cases, all remaining integers are fixed points. Any signed permutation is a product of cycles and expressing a given σ in this way is called cycle notation of σ. For even-signed permutations and ordinary permutations, the cycle notation is defined analogously. We illustrate the cycle notation by some examples:

$$\begin{aligned}
[4,2,6,5,1,3] &= (1,4,5)(3,6), \\
[4,2,-6,5,1,-3] &= (1,4,5)(3,-6), \\
[4,2,-6,5,1,3] &= (1,4,5)(3,-6,-3) = (1,4,5)(6,3,-6), \\
[4,2,6,5,-1,-3] &= (1,4,5,-1)(3,6,-3).
\end{aligned}$$

Define the length of a cycle $c = (i_1, \ldots, i_k)$ to be $k-1$ and denote it by $\mathrm{l}(c)$. Then the absolute length of a signed permutation σ is equal to the sum of the lengths of the cycles in its cycle notation $\sigma = c_1 \cdots c_k$, in symbols

$$\mathrm{l}_T(\sigma) = \mathrm{l}(c_1) + \ldots + \mathrm{l}(c_k).$$

The following definition goes back mainly to work by Brady and Watt [11, 12]:

Definition 1.27. Let c be a Coxeter element in W. The non-crossing partition lattice $NC(W,c)$ is defined as the interval in $\mathsf{Abs}(W)$ between 1 and c,

$$NC(W,c) := [1,c]_T = \big\{\omega \in W : 1 \leq_T \omega \leq_T c\big\}.$$

This definition seems to depend on the choice of the Coxeter element c, but as all Coxeter elements form a conjugacy class in W and since conjugation by a group element is an automorphism on $\mathsf{Abs}(W)$ it follows that for Coxeter elements c and c',

$$NC(W, c) \cong NC(W, c').$$

In [13], Brady and Watt proved that $NC(W, c)$ is in fact a lattice. Furthermore, Armstrong showed in [4] that it is self-dual and locally self-dual.

We now want to describe how non-crossing set partitions can be seen as the type A instance of non-crossing partitions. Write $NC(A_{n-1})$ for $NC(A_{n-1}, c)$ where $c = (1, 2, \ldots, n)$. Then

$$\sigma \leq_T c \Leftrightarrow \text{all cycles in } \sigma \text{ are increasing and pairwise non-crossing},$$

where a cycle $(i_1, \ldots, i_k)$ is called increasing if $i_1 < \ldots < i_k$ and where two cycles $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_{k'})$ are called non-crossing if the associated blocks $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{k'}\}$ do not cross in the sense of Example 1.2. This yields a bijection between non-crossing set partitions and $NC(A_{n-1})$: map some $\{B_1, \ldots, B_k\}$ to the permutation having a cycle for each B_i with the given elements in increasing order. Furthermore, the ordering of non-crossing set partitions by refinement turns $NC(n)$ into a lattice with $\hat{1} = [n]$ and $\hat{0} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and we have

$$NC(n) \cong NC(A_{n-1}).$$

There exists also a description of non-crossing partitions of types B and D in terms of set partitions. They were introduced in type B by Reiner in [29] and in type D by Athanasiadis and Reiner in [6]. As we only need the description in type B, we relax the description in type D.

Definition 1.28. A type B_n set partition is a set partition $\mathcal{B}$ of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ satisfying the following two conditions:

(i) if B is a block in $\mathcal{B}$ then $-B$ is also a block in $\mathcal{B}$,
(ii) there exists at most one block B in $\mathcal{B}$ for which $B = -B$.

Order the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ by

$$-1 < -2 < \ldots < -n < 1 < 2 < \ldots < n.$$

A type B_n set partition $\{B_1, \ldots, B_k\}$ is called non-crossing if

$$a < b < c < d \text{ such that } a, c \in B_i, b, d \in B_j \text{ implies } i = j.$$

The lattice of all non-crossing set partitions of type B_n ordered by refinement is denoted by $NC_B(n)$.

Note. One can visualize non-crossing set partitions of type B_n in the same way as non-crossing set partitions of type A_{n-1}. This visualization shows immediately that the property of being non-crossing implies (ii) in the definition of type B_n set partitions.

This yields a bijection between $NC_B(n)$ and $NC(B_n)$ where $NC(B_n) := NC(B_n, c)$ with $c = (1, 2, \ldots, n, -1)$: map some $\{B_1, \ldots, B_k\}$ to the permutation having a cycle for each B_i with the given elements in increasing order. We have

$$NC_B(n) \cong NC(B_n).$$

1.5. **Coxeter sortable elements.** In [27], N. Reading introduced another subset of a reflection group W which he called Coxeter sortable elements and gave a bijection to non-crossing partitions. Let c be a Coxeter element in W and fix a reduced word for c, say $c = s_1 s_2 \cdots s_l$. For $\omega \in W$, the c-sorting word of ω is defined to be the lexicographically first reduced expression for ω when expressed as a subword of the half infinite word

$$c^\infty := s_1 s_2 \cdots s_l | s_1 s_2 \cdots s_l | s_1 s_2 \cdots s_l | \cdots ,$$

where the divider $|$ is introduced just to distinguish between different occurrences of $s_1 s_2 \cdots s_l$.

Note. The c-sorting word can be interpreted as a sequence of subsets of the simple reflections: the subsets in this sequence are the sets of letters of the c-sorting word which occur between adjacent dividers.

Definition 1.29. An element $\omega \in W$ is called c-sortable if its c-sorting word defines a sequence of subsets which is decreasing under inclusion. Furthermore, define $\mathrm{Cox}_c(W)$ as the set of all c-sortable elements in W,

$$\mathrm{Cox}_c(W) := \big\{\omega \in W : \omega \ c\text{-sortable}\big\}.$$

Note. The definition of c-sortable does not depend on the specific choice of the reduced word for c as different reduced words are related by commutations of letters with no commutations across dividers.

Example 1.30. Consider the case of the symmetric group $\mathcal{S}_n$. Let c be the long cycle $(n, \ldots, 2, 1) = s_{n-1} \cdots s_2 s_1$. Then we have for $\sigma \in \mathcal{S}_n$,

$$\sigma \text{ is } c\text{-sortable} \Leftrightarrow \sigma \text{ is 231-avoiding.}$$

We consider only the case $n = 3$, the case $n > 3$ is similar. When expressing the permutations in $\mathcal{S}_3$ by their c-sorting words, we have

$$\mathcal{S}_3 = \big\{1, s_2, s_2 s_1, s_2 s_1 | s_2, s_1, s_1 | s_2\big\}.$$

The only element which is *not* Coxeter sortable is $s_1 | s_2 = [2, 3, 1]$.

As already mentioned, Reading proved the following theorem bijectively, see [27, Section 6]:

Theorem 1.31 (Reading)**.** *Let W be a real reflection group and let c be a Coxeter element in W. Then*

$$|\mathrm{Cox}_c(W)| = \mathrm{Cat}(W).$$

Remark. In [27], Reading showed moreover that Coxeter sortable elements provide a connection between non-crossing partitions and *clusters* (facets) in the *cluster complex*.

2. A bijection between non-nesting and non-crossing partitions

In this section, we describe the q-Catalan numbers $\mathrm{Cat}(W; q)$ as well as the q-Catalan numbers q-$\mathrm{Cat}(W; q)$ in terms of non-crossing partitions, when W is either of type A or of type B. By definition,

$$\begin{aligned} \mathrm{Cat}(W; q) &= \sum q^{\mathrm{area}(D)}, \\ \text{q-}\mathrm{Cat}(W; q) &= \sum q^{\mathrm{maj}(D)}, \end{aligned}$$

where the sums range over all Dyck paths of the given type and where area and maj are the appropriate statistics.

Let rev be the involution on signed permutations which reverses the negative elements in the one-line notation, e.g. $\mathrm{rev}([2,-4,3,-1]) = [2,-1,3,-4]$. We will bijectively prove the following theorem:

Theorem 2.1. *Let W be the reflection group A_{n-1} or B_n. Then the q-Catalan numbers* $\mathrm{Cat}(W;q)$ *and* $\text{q-Cat}(W;q)$ *can be interpreted in terms of non-crossing partitions as follows:*

$$\mathrm{Cat}(W;q) = \sum_{\sigma \in \mathrm{rev}(NC(W))} q^{\mathrm{ls}(\sigma)}, \tag{2}$$

$$\text{q-Cat}(W;q) = \sum_{\sigma \in \mathrm{rev}(NC(W))} q^{\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma)}. \tag{3}$$

Remark. The analogous statement is false in type D: by computer experiments it is easy to show that for any Coxeter element $c \in D_4$,

$$\mathrm{Cat}(D_4;q) \neq \sum_{\sigma \in \mathrm{rev}(NC(D_4,c))} q^{\mathrm{ls}(\sigma)}.$$

2.1. **The bijection in type A.** Define a map $\phi_n : NN(A_{n-1}) \rightarrow NC(A_{n-1})$ as indicated in Figure 7 on page 17: write the numbers 1 to n below the root poset of type A_{n-1} from right to left and then associate to a given order ideal $I \trianglelefteq \Phi^+$ the permutation obtained by the shown "shelling" of I. In terms of the root poset Φ^+, ϕ_n can be described as follows: for $1 \leq i < j \leq n$, set $[i,j]$ to be the positive root $\epsilon_j - \epsilon_i$. For an order ideal $I = \{[i_1,j_1],\ldots,[i_{max},j_{max}]\} \trianglelefteq \Phi^+$, let $[a_1,b_1],\ldots,[a_k,b_k]$ with $a_1 < \ldots < a_k$ be the maximal elements in I. The list of maximal elements in I decomposes into blocks, such that $[a_{i-1},b_{i-1}]$ is the last element in one block and $[a_i,b_i]$ is the first element in the next block if and only if $b_{i-1} \leq a_i$. To I, we associate a permutation $\sigma(I)$ having a cycle for each block, where the cycle starts with the first a_i in the block followed by all a_i's that are equal to b_{i-1}'s and which ends with the last b_i in the block.
Furthermore, set $I' \trianglelefteq \Phi^+$ to be the order ideal given by

$$I' := \big\{[i+1,j-1] : [i,j] \in I \text{ and } j-i > 2\big\}.$$

Then ϕ_n can be described as

$$\phi_n(I) := \sigma(I) \circ \phi_n(I').$$

Example 2.2. The ideal shown in Figure 7 is given by

$$I = \left\{ \begin{array}{c} [1,2],[2,3],[3,4],[4,5],[5,6],[6,7],[7,8],[8,9], \\ [1,3],[2,4],[3,5],[4,6],[5,7],[7,9],[1,4],[2,5],[3,6] \end{array} \right\},$$

its maximal elements are $\{[1,4],[2,5],[3,6],[5,7],[7,9]\}$ and $I' = \{[2,3],[3,4],[4,5]\}$. This gives $\phi_9(I) = (1,7,9) \circ \phi_9(I')$ and as all elements in I' are minimal, we have $\phi_9(I') = (2,3,4,5)$ and thereby

$$\phi_9(I) = (1,7,9)(2,3,4,5).$$

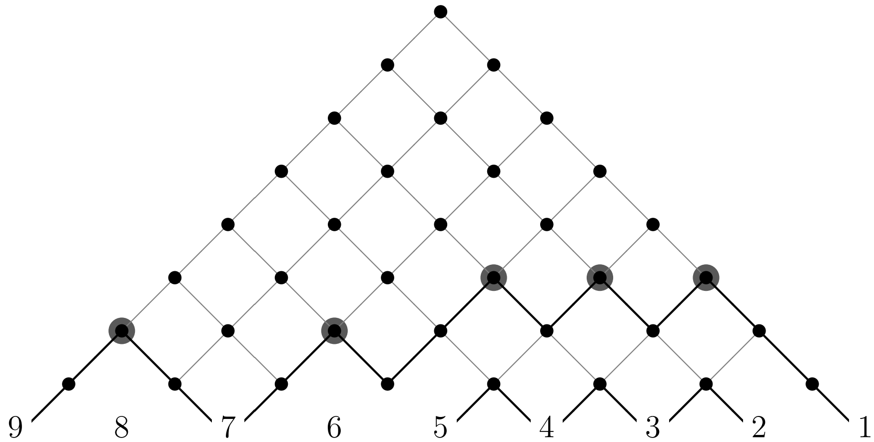


FIGURE 7. The bijection ϕ_9 sending the shown non-nesting partition to the non-crossing partition $\sigma = (1, 7, 9)(2, 3, 4, 5) = [7, 3, 4, 5, 2, 6, 9, 8, 1]$.

From the construction, it is clear that ϕ_n is in fact a bijection between $NN(A_{n-1})$ and $NC(A_{n-1})$.

For $\sigma \in \mathcal{S}_n$, $\mathrm{l}_S(\sigma) = \mathrm{inv}(\sigma)$. Therefore, the following proposition proves Eq. (2) in Theorem 2.1 for type A:

Proposition 2.3. *ϕ_n maps the area statistic on $NN(n)$ to the inversion number on $NC(A_{n-1})$, i.e., for $I \in NN(A_{n-1})$, we have*

$$\mathrm{area}(I) = |I| = \mathrm{inv}([\sigma_1\sigma_2 \cdots \sigma_n]),$$

where $[\sigma_1\sigma_2 \cdots \sigma_n]$ is the one-line notation of $\phi_n(I)$.

Proof. For simplicity, we assume that a given I contains only one "shell" or, equivalently, $\phi_n(I)$ is a 1-cycle, say $(i_1, \ldots, i_k)$, the general case is solved by applying the same argument several times. The number of elements in I is equal to $2(i_k - i_1) - 1 - (k-2) = 2(i_k - i_1) - k + 1$. It is easy to see that this is also equal to the inversion number of the cycle $(i_1, \ldots, i_k)$. ☐

Example (continued) 2.4. Let I and $\sigma = \phi_9(I)$ as in Example 2.2. Then the "first shell" contains 14 elements which is equal to the inversion number of the associated cycle $(1, 7, 9) = [7, 2, 3, 4, 5, 6, 9, 8, 1]$, the "second shell" contains 3 elements which is equal to the inversion number of the associated cycle $(2, 3, 4, 5) = [1, 3, 4, 5, 2, 6, 7, 8, 9]$.

The following theorem together with the fact that the major index on Dyck paths is symmetric proves Eq. (3) in Theorem 2.1 for type A:

Theorem 2.5. *Let $I \trianglelefteq \Phi^+$ be an order ideal in the root poset of type A_{n-1} and let $N = \binom{n}{2}$ be the number of positive roots. Then*

$$\mathrm{maj}(I) + \mathrm{maj}(\phi_n(I)) + \mathrm{imaj}(\phi_n(I)) = 2N = n(n-1).$$

Before proving the theorem, we get back to the ongoing example:

Example (continued) 2.6. The descent set of the ideal I considered in Example 2.2 and the descent set and the inverse descent set of $\sigma = \phi_9(I)$ are given by

$$\mathrm{Des}(I) = \{5, 8, 11, 13\},\ \mathrm{Des}(\sigma) = \{1, 4, 7, 8\},\ \mathrm{iDes}(\sigma) = \{1, 2, 6, 8\}$$

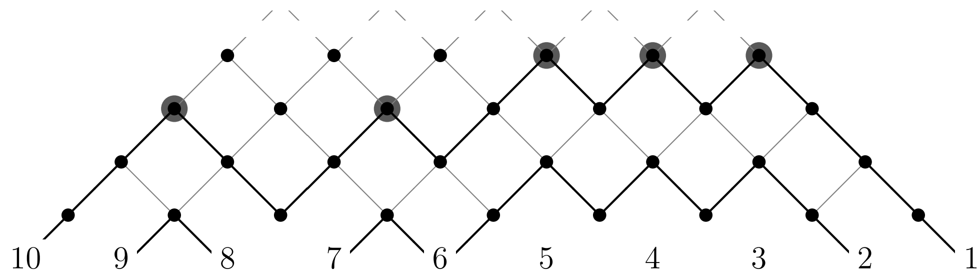


FIGURE 8. The lifting $\Delta(I)$ of the non-nesting partition I shown in Figure 7 and its image $\phi_{10}(\Delta(I)) = (1,10)(2,6,7)(8,9) = [10,6,3,4,5,7,2,9,8,1]$.

and therefore,

$$\begin{aligned}
\mathrm{maj}(I) &= (18-5)+(18-8)+(18-11)+(18-13) = 35, \\
\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma) &= (1+4+7+8)+(1+2+6+8) = 20+17, \\
\mathrm{maj}(I)+\mathrm{maj}(\sigma)+\mathrm{imaj}(\sigma) &= 35+37 = 72 = 9\cdot 8 = n(n-1).
\end{aligned}$$

We prove the theorem in several steps:

Lemma 2.7. *Let $\sigma \in NC(n)$. Then* $\mathrm{des}(\sigma) = \mathrm{ides}(\sigma)$.

Proof. We first prove the lemma for the case that σ has only one cycle: let $\sigma = (i_1,\dots,i_k)$. As σ is in $NC(n)$, we have $i_1 < \ldots < i_k$. Therefore, we can describe the descent set and the inverse descent set of σ:

$$\begin{aligned}
\mathrm{Des}(\sigma) &= \{i_l : l < k, i_l + 1 < i_{l+1}\} \cup \{i_k - 1\}, \\
\mathrm{iDes}(\sigma) &= \{i_l - 1 : 1 < l, i_{l-1} + 1 < i_l\} \cup \{i_1\}.
\end{aligned}$$

The case that σ has more than one cycle follows from the fact that σ is non-crossing and therefore the descent set and the inverse descent set of σ are given by the above rule for each cycle. $\square$

Lemma 2.8. *Let $I \in NN(n)$. Then*

$$\mathrm{des}(I) + \mathrm{des}(\phi_n(I)) = n - 1.$$

Proof. Let $\sigma := \phi_n(I)$ and let *min* respectively *max* be the minimal respectively maximal element not mapped by σ to itself. Set a to be the number of valleys of I between positions *min* and *max*. Then the proof of the previous lemma implies $\mathrm{des}(\sigma) = max - min - a$. By definition, $\mathrm{des}(I)$ equals the total number of valleys of I and therefore, $\mathrm{des}(I) = a + min - 1 + n - max$. This completes the proof. $\square$

Define a lifting Δ from $NN(n)$ to $NN(n+1)$ by taking an order ideal $I \in NN(n)$ and embed it into $NN(n+1)$ by adding the whole "bottom row", see Figure 8 for an example.

Lemma 2.9. *Let $I \in NN(n)$ and $\sigma := \phi_n(I)$. Furthermore, set $I' := \Delta(I)$ and $\sigma' := \phi_{n+1}(\Delta(I))$. Then*

$$\mathrm{maj}(I) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) + 2n = \mathrm{maj}(I') + \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma').$$

Proof. Observe that

$$\mathrm{Des}(\sigma') = \mathrm{iDes}(\sigma) \cup \{n\} \quad , \quad \mathrm{iDes}(\sigma') = \{i+1 : i \in \mathrm{Des}(\sigma)\} \cup \{1\}$$

and therefore,

$$\mathrm{maj}(\sigma') - \mathrm{imaj}(\sigma) = n \quad , \quad \mathrm{imaj}(\sigma') - \mathrm{maj}(\sigma) = \mathrm{des}(\sigma) + 1. \tag{4}$$

On the other hand, we have

$$\mathrm{maj}(I') - \mathrm{maj}(I) = \mathrm{des}(I) = \mathrm{des}(I'). \tag{5}$$

The Lemma follows by (4), (5) and Lemma 2.8. □

Example (continued) 2.10. In our ongoing example, we have already seen that

$$\mathrm{maj}(I) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) + 2n = 72 + 18 = 90 = n(n+1).$$

On the other hand, we have

$$\mathrm{Des}(I') = \{6, 9, 12, 14\}, \mathrm{Des}(\sigma') = \{1, 2, 6, 8, 9\}, \mathrm{iDes}(\sigma) = \{1, 2, 5, 8, 9\}.$$

This gives

$$\begin{aligned} \mathrm{maj}(I') &= (20-6) + (20-9) + (20-12) + (20-14) = 39, \\ \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma') &= (1+2+6+8+9) + (1+2+5+8+9) = 26 + 25, \\ \mathrm{maj}(I') + \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma') &= 39 + 51 = 90 = n(n+1). \end{aligned}$$

Proof of Theorem 2.5. We prove the Theorem by induction on n. Let $I \in NN(n)$ and $I' \in NN(n')$ and let D and D' be the associated Dyck paths in $\mathcal{D}_n$ and $\mathcal{D}_{n'}$ respectively. Then the concatenation of I and I' is given by the concatenation $DD' \in \mathcal{D}_{n+n'}$. The proof consists of two parts:

(i) first, we prove that if the theorem holds for elements $I \in NN(n)$ and $I' \in NN(n')$ then it holds for the concatenation II' of I and I' which lies in $NN(n+n')$, and
(ii) second, we prove that if the theorem holds for $I \in NN(n)$ then it holds also for $\Delta(I) \in NN(n+1)$.

As the case $n = 1$ is obvious, the theorem then follows.

(i) set $\sigma := \phi_n(I), \sigma' := \phi_{n'}(I')$ and $\tau := \phi_{n+n'}(II')$. Then we have

$$\begin{aligned} \mathrm{maj}(II') + \mathrm{maj}(\tau) + \mathrm{imaj}(\tau) &= \mathrm{maj}(I) + \mathrm{maj}(I') + 2n(\mathrm{des}(I') + 1) \\ &+ \mathrm{maj}(\sigma) + \mathrm{maj}(\sigma') + n\,\mathrm{des}(\sigma') \\ &+ \mathrm{imaj}(\sigma) + \mathrm{imaj}(\sigma') + n\,\mathrm{ides}(\sigma'). \end{aligned}$$

By Lemma 2.7 and Lemma 2.8, the right-hand side of this equation is equal to $\mathrm{maj}(I) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) + \mathrm{maj}(I') + \mathrm{maj}(\sigma') + \mathrm{imaj}(\sigma') + 2nn'$. By induction, this reduces to $n(n-1) + n'(n'-1) + 2nn' = (n+n')(n+n'-1)$.
(ii) this is an immediate consequence of Lemma 2.9.

□

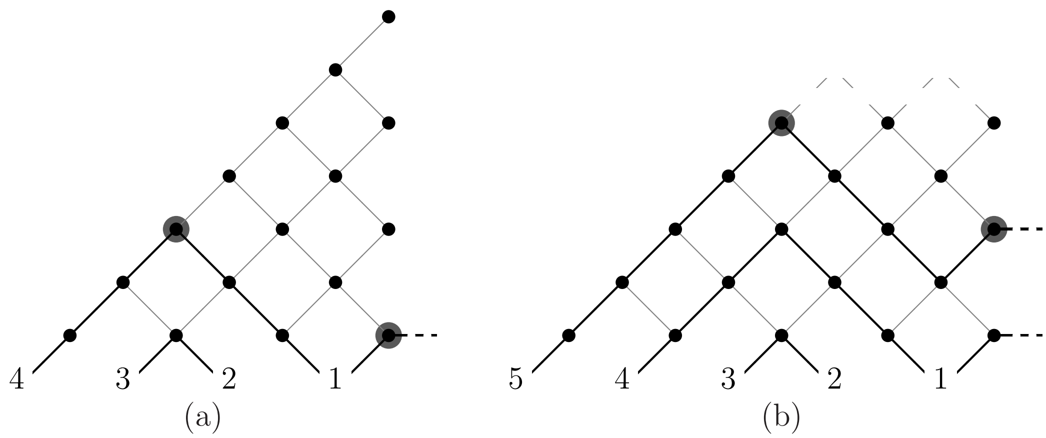


FIGURE 9. (a) shows the bijection ϕ_4 sending the given non-nesting partition I to the non-crossing partition $\phi_4(I) = (1, 4, -1)(2, 3) = [4, 3, 2, -1]$, (b) shows its lift $\Delta(I)$, as shown, the image of this lift is $\phi_5(\Delta(I)) = (1, 4, -1)(2, 3)(5, -5) = [4, 3, 2, -1, -5]$.

2.2. **The bijection in type B.** The bijection $\phi_n : NN(A_{n-1}) \longrightarrow NC(A_{n-1})$ can be adapted to type B_n as follows: write the numbers 1 to n below the root poset of type B_n from right to left as shown in Figure 9 and map a given order ideal $I \trianglelefteq \Phi^+$ in the same way to a signed permutation as in type A_{n-1} with the additional rule that if a "shell" ends at the "right boundary" of Φ^+ then add the negative of the first element of the given cycle to its end. This map defines a bijection between $NN(B_n)$ and $\mathrm{rev}(NC(B_n))$. To prove Theorem 2.1 in type B we only have to modify the lifting Δ from $NN(B_n)$ to $NN(B_{n+1})$ which is now defined by adding *two* "bottom rows", see Figure 9 for an example, and slightly different induction steps.

For $\sigma \in B_n$, $\mathrm{l}_S(\sigma) = \mathrm{inv}([\sigma_1\sigma_2 \cdots \sigma_n]) - \sum_{i \in \mathrm{Neg}} \sigma(i)$. Therefore, the following proposition proves Eq. (2) in Theorem 2.1 for type B:

Proposition 2.11. *For $I \in NN(B_n)$, we have*

$$\mathrm{area}(I) = |I| = \mathrm{inv}([\sigma_1\sigma_2 \cdots \sigma_n]) - \sum_{i \in \mathrm{Neg}(\sigma)} \sigma_i,$$

where $\sigma := \phi_n(I)$ and $[\sigma_1\sigma_2 \cdots \sigma_n]$ is its one-line notation.

Proof. The proof follows exactly the same idea as the proof in type A. □

The following theorem together with the fact that the major index of Dyck paths of type B is symmetric proves Eq. (3) in Theorem 2.1 for type B:

Theorem 2.12. *Let $I \trianglelefteq \Phi^+$ be an order ideal in the root poset of type B_n and let $N = n^2$ be the number of positive roots. Then*

$$\mathrm{maj}(I) + \mathrm{maj}(\phi_n(I)) + \mathrm{imaj}(\phi_n(I)) = 2N = 2n^2.$$

Proof. We prove the theorem as in type A by induction. As in type A, the case $n = 1$ is obvious, therefore the theorem follows by proving the following 3 cases:

(i) first, we prove that if the theorem holds for elements $I \in NN(A_{n-1})$ and $I' \in NN(B_{n'})$ then it holds for the concatenation $II' \in NN(B_{n+n'})$,

(ii) second, we prove that if the theorem holds for elements $I \in NN(A_{n-1})$ and $I' \in NN(B_{n'})$ then it holds for the order ideal $J \in NN(B_{n+n'})$ obtained by replacing the last east step in the Dyck word associated to I by a north step and then concatenating it with I', and

(iii) third, we prove that if the theorem holds for $I \in NN(B_n)$ then it holds also for $\Delta(I) \in NN(B_{n+1})$.

Set $\sigma := \phi_n(I), \sigma' := \phi_{n'}(I')$ and $\tau := \phi_{n+n'}(II')$

(i) the proof of (i) is the same as in type A_{n-1} with n replaced by $2n$:

$$\begin{aligned}\operatorname{maj}(II') + \operatorname{maj}(\tau) + \operatorname{imaj}(\tau) &= \operatorname{maj}(I) + \operatorname{maj}(I') + 4n(\operatorname{des}(I') + 1)\\ &+ \operatorname{maj}(\sigma) + \operatorname{maj}(\sigma') + 2n\operatorname{des}(\sigma')\\ &+ \operatorname{imaj}(\sigma) + \operatorname{imaj}(\sigma') + 2n\operatorname{ides}(\sigma').\end{aligned}$$

By Lemma 2.7 and Lemma 2.8, the right-hand side of this equation is equal to $\operatorname{maj}(I)+\operatorname{maj}(\sigma)+\operatorname{imaj}(\sigma)+\operatorname{maj}(I')+\operatorname{maj}(\sigma')+\operatorname{imaj}(\sigma')+4nn'$. By induction, this reduces to $2n^2 + 2n'^2 + 4nn' = 2(n + n')^2$.

(ii)

$$\begin{aligned}\operatorname{maj}(II') + \operatorname{maj}(\tau) + \operatorname{imaj}(\tau) &= \operatorname{maj}(I) + \operatorname{maj}(I') + 4n(\operatorname{des}(I') + 1)\\ &+ \operatorname{maj}(\sigma) + \operatorname{maj}(\sigma') + 2n\operatorname{des}(\sigma') + 1\\ &+ \operatorname{imaj}(\sigma) + \operatorname{imaj}(\sigma') + 2n\operatorname{ides}(\sigma') + 1 - 2.\end{aligned}$$

Again, by Lemma 2.7 and Lemma 2.8, the right-hand side of this equation is equal to $\operatorname{maj}(I) + \operatorname{maj}(\sigma) + \operatorname{imaj}(\sigma) + \operatorname{maj}(I') + \operatorname{maj}(\sigma') + \operatorname{imaj}(\sigma') + 4nn' = 2n^2 + 2n'^2 + 4nn' = 2(n + n')^2$.

(iii) We have $\phi_{n+1}(\Delta(I)) = \sigma \circ (n + 1, -n - 1)$. As $\operatorname{maj}(\Delta(I)) = \operatorname{maj}(I)$, this gives

$$\operatorname{maj}(\Delta(I))+\operatorname{maj}(\phi_{n+1}(\Delta(I)))+\operatorname{imaj}(\phi_{n+1}(\Delta(I))) = \operatorname{maj}(I)+\operatorname{maj}(\sigma)+\operatorname{imaj}(\sigma)+4n+2,$$

and the right-hand side is by induction equal to $2n^2 + 4n + 2 = 2(n + 1)^2$.

□

Remark. In Example 1.3, we mentioned a bijection between non-crossing and non-nesting set partitions that locally converts each nesting into a crossing. When generalizing this bijection to type B in the canonical way, one obtains – as we do – a bijection between $NN(B_n)$ and $\operatorname{rev}(NC(B_n))$.

3. A bijection between non-nesting partitions and Coxeter sortable elements

Now, we want to describe the q-Catalan numbers $\operatorname{Cat}(W; q)$ and q-$\operatorname{Cat}(W; q)$ in terms of Coxeter sortable elements for W being of type A or B.

Surprisingly, for Coxeter sortable elements we obtain almost the same as for non-crossing partitions with the advantage that we do not need to reverse the negative elements, compare Theorem 2.1.

Theorem 3.1. *Let W be the reflection group A_{n-1} or B_n and set $c = s_{n-1} \cdots s_2 s_1$ or $c = s_{n-1} \cdots s_2 s_1 s_0$ respectively. Then the q-Catalan numbers* $\operatorname{Cat}(W; q)$ *and* q-$\operatorname{Cat}(W; q)$

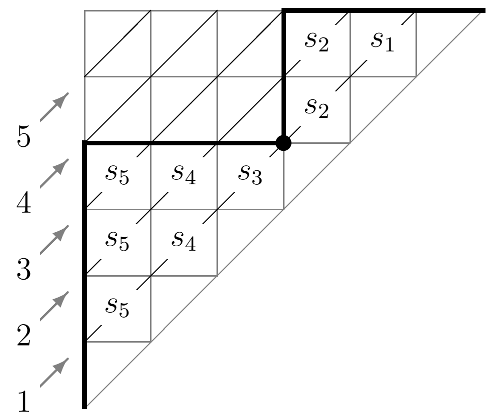


FIGURE 10. A Dyck path of length 6 with cells labelled by simple transpositions.

can be interpreted in terms of c-sortable elements as

$$\operatorname{Cat}(W;q) = \sum_{\operatorname{Cox}_c(W)} q^{\operatorname{l}_S(\sigma)}, \tag{6}$$

$$\text{q-Cat}(W;q) = \sum_{\operatorname{Cox}_c(W)} q^{\operatorname{maj}(\sigma)+\operatorname{imaj}(\sigma)}, \tag{7}$$

where the sums range over all c-sortable elements of the given type and where maj *and* imaj *are the appropriate statistics.*

Remark. As for non-crossing partitions, the analogous statement is false in type D: for any Coxeter element $c \in D_4$,

$$\operatorname{Cat}(D_4;q) \neq \sum_{\sigma\in\operatorname{Cox}(D_4,c)} q^{\operatorname{l}_S(\sigma)}.$$

3.1. **The bijection in type A.** In Example 1.30, we have seen that for $W = A_{n-1}$ and $c = (n,\dots,2,1)$, being c-sortable is the same as being 231-avoiding. Bijections between 3-pattern-avoiding permutations and Dyck paths are very well studied, e.g. see [7, 15, 25, 28], this connection will be explored in the proof of Proposition 3.3 and [33, Section 5].

Let D be a Dyck path of length n and identify D with the set $\{b_{ij}\}$ of cells below D as described in the beginning of Section 1.2. Label every cell b_{ij} by s_{n-1-i}. The bijection between Dyck paths and c-sortable elements is then defined by mapping $D \in \mathcal{D}_n$ to the c-sorting word $\sigma := \prod s_{n-1-i}$, where the product ranges over all cells b_{ij} in the order as indicated in Figure 10. By construction, σ is c-sortable.

Example 3.2. The Dyck path shown in Figure 10 is mapped to the c-sortable element

$$s_5s_4s_3s_2s_1|s_5s_4s_2|s_5 = [6,2,1,5,4,3].$$

Theorem 3.1 follows in type A from the following proposition:

Proposition 3.3. *Let D be a Dyck path and let σ be the image of D under the bijection just defined. Then* $\mathrm{area}(D) = \mathrm{l}_S(\sigma)$ *and furthermore,*

$$\begin{aligned}\mathrm{Des}(\sigma) &= [n-1] \setminus \{n-i : i \in \mathrm{Set}_X(D)\},\\ \mathrm{iDes}(\sigma) &= [n-1] \setminus \{n-i : i \in \mathrm{Set}_Y(D)\},\end{aligned}$$

in particular,

$$\mathrm{maj}(D) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) = n(n-1).$$

Proof. The fact that $\mathrm{area}(D) = \mathrm{l}_S(\sigma)$ follows directly from the construction. Denote the bijection in the proposition by γ, then the bijection β between $\mathcal{S}_n(231)$ and $\mathcal{D}_n$ defined by J. Bandlow and K. Killpatrick in [7] can be described in terms of γ by

$$\beta = \mathbf{c} \circ \gamma^{-1}.$$

In other words, β^{-1} maps a given Dyck path D to the image of the conjugate of D under γ. The proposition then follows with [33, Theorem 3.12] and the description of β in [33, Section 5]. □

The given bijection also preserves another statistic on Dyck paths, namely the length of the last descent, we will use this fact for constructing the bijection in type B:

Proposition 3.4. *Let D be a Dyck path of length n and let k be the number of east steps after the last north step. Then $\sigma(k) = 1$ for σ being the image of D and furthermore, $\{1, \ldots, k-1\} \subseteq \mathrm{Des}(\sigma)$.*

Proof. Let $S_1|S_2|\cdots|S_k$ be the initial segment of the the c-sorting word for σ, with S_k possibly empty. Then, by construction, the last simple reflection in S_i is s_i for $i < k$ and s_k is *not* contained in S_k. Therefore, k is mapped by σ to 1 and, as σ is 231-avoiding, it follows immediately that $\{1, \ldots, k-1\} \subseteq \mathrm{Des}(\sigma)$ (of course, the later can also be obtained directly). □

3.2. **The bijection in type B.** As in type A, let D be a Dyck path of type B_n and identify D with the set $\{b_{ij}\}$ of cells below D as described in Definition 1.12. Label every cell b_{ij} with $j < n$ by s_{n-1-i} and b_{ij} with $j \geq n$ by $s_{2(n-1)-(i+j)}$. The bijection between Dyck paths of type B_n and c-sortable elements is then defined by mapping $D \in \mathcal{D}_n$ to the c-sorting word σ which is the product of the simple transpositions in the cells b_{ij} in the order as indicated in Figure 11.

Example 3.5. The Dyck path shown in Figure 10 is mapped to the c-sortable element

$$s_5s_4s_3s_2s_1s_0|s_5s_4s_2s_1s_0|s_5s_2s_1 = [1, -2, -6, 5, 4, 3].$$

To see that the image $\sigma = S_1|S_2|\cdots|S_k$ of a given D is in fact c-sortable, we only have to show that $S_1|S_2|\cdots|S_k$ is a reduced expression for σ as the inclusion property $S_1 \supseteq S_2 \supseteq \ldots \supseteq S_k$ is given by construction.

Proposition 3.6. *$S_1|S_2|\cdots|S_k$ is a reduced expression for σ.*

Proof. If s_is_{i-1} occurs in S_j and in S_{j+1} for some i and j then s_{i-2} occurs also in S_j except for the case $i = 1$. But if s_1s_0 occurs in S_j and in S_{j+1} and furthermore, s_1 occurs in S_{j+2} then s_2 occurs in S_{j+2} left of s_1. The proposition follows. □

This proposition immediately implies the following corollary which proves Eq. (6) in Theorem 3.1 for type B:

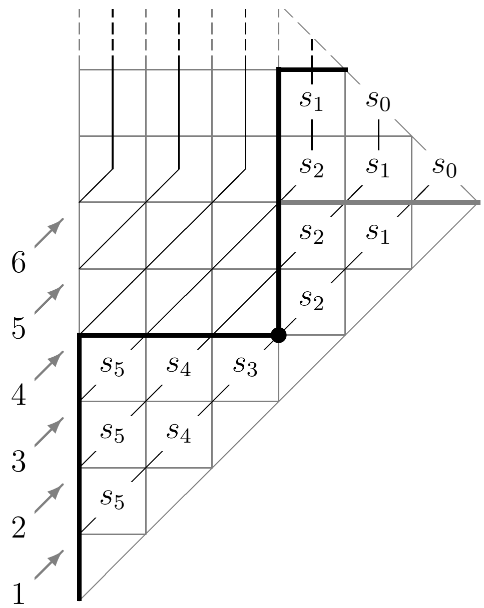


FIGURE 11. A Dyck path of type B_6 with cells labelled by simple transpositions.

Corollary 3.7. *Let D be a Dyck path of type B_n and let σ be its image under the above bijection. Then*

$$\mathrm{area}(D) = \mathrm{l}_S(\sigma).$$

To prove Eq. (7) in Theorem 3.1 for type B, we use the fact that a Dyck path D of type B_n consists of a "lower part" D_1 which is a Dyck path of type A_{n-1}, and an "upper part" D_2. D_1 is obtained from D by replacing all north steps after the n-th north step by east steps and D_2 is obtained as the suffix of D after the n-th north step. For example, the Dyck path of type B_6 in Figure 11 consists of a lower part which is the Dyck path shown in Figure 10 and an upper part given by the word NNE.

As s_i and s_j commute for $|i-j|>1$, we can write the image of D as the image of D_1 followed by the product of the cells below D_2 row by row from bottom to top and from right to left. Set σ, σ_1 and σ_2 to be the signed permutations associated to D, D_1 and D_2. For example,

$$\begin{aligned}\sigma &= s_5s_4s_3s_2s_1s_0|s_5s_4s_2s_1s_0|s_5s_2s_1 \\ &= \sigma_1 \cdot \sigma_2 \\ &= s_5s_4s_3s_2s_1|s_5s_4s_2|s_5 \cdot s_0s_1s_2|s_0s_1.\end{aligned}$$

As we have seen in the previous section, we have, when considering D_1 in type A_{n-1},

$$\mathrm{maj}(D_1) + \mathrm{maj}(\sigma_1) + \mathrm{imaj}(\sigma_1) = n(n-1).$$

This gives, when considered in type B_n,

$$\mathrm{maj}(D_1) + \mathrm{maj}(\sigma_1) + \mathrm{imaj}(\sigma_1) = 2n(n-1) + 2n = 2n^2.$$

We will use this fact and are going to show that

$$\mathrm{maj}(D) + \mathrm{maj}(\sigma) + \mathrm{imaj}(\sigma) = \mathrm{maj}(D_1) + \mathrm{maj}(\sigma_1) + \mathrm{imaj}(\sigma_1), \tag{8}$$

Eq. (7) in Theorem 3.1 for type B then follows.

Note. As we use the result in type A to prove type B, so far everything is only proved modulo Proposition 3.3.

We proof Eq. (8) in several steps:

Lemma 3.8. $\mathrm{neg}(D) + \mathrm{neg}(\sigma) = n$.

Proof. By definition, $\mathrm{neg}(D)$ is given by the number of east steps in D and by construction of σ, the number of s_0's in the c-sorting word for σ in $n - \mathrm{neg}(D)$. Therefore,

$$\mathrm{neg}(\sigma) = n - \mathrm{neg}(D).$$

□

To keep the notation simple, we set

$$\mathrm{maj}(D_2) := \sum_{i \in \mathrm{Des}(D_2)} 2(k-i),$$

where k is the number of steps in D_2.

Lemma 3.9. *Let k be the number of steps in D_2 or, equivalently, let k be the number of east steps in D_1 after the last north step. Then*

$$\mathrm{maj}(D) = \mathrm{maj}(D_1) + \mathrm{maj}(D_2) - 2(n - \mathrm{neg}(D)).$$

Proof. By definition, $\mathrm{neg}(D)$ and $\mathrm{neg}(D_1)$ are the number of east steps in D and in D_1, in particular, $\mathrm{neg}(D_1) = n$, and we have

$$\mathrm{maj}(D) = 2 \cdot \Big(\mathrm{neg}(D) + \sum_{i \in \mathrm{Des}(D)} (2n-i) \Big) \quad , \quad \mathrm{maj}(D_1) = 2 \cdot \Big(n + \sum_{\substack{i \in \mathrm{Des}(D) \\ i < 2n-k}} (2n-i) \Big).$$

The lemma follows. □

Lemma 3.10. *Let $S_1|S_2|\cdots|S_k$ be the expression for σ_2. Then*

$$\mathrm{Neg}(\sigma_2) = \{ j+1 : s_j \text{ is the rightmost simple reflection in } S_i \text{ for some } i \}$$

and the images of $\mathrm{Neg}(\sigma_2)$ under σ_2 are the negatives of the first $\mathrm{neg}(\sigma_2)$ integers in increasing order and the image of the complement of $\mathrm{Neg}(\sigma_2)$ are the last $k - \mathrm{neg}(\sigma_2)$ integers also in increasing order.

Proof. This can be seen immediately from the expression $\sigma_2 = S_1|S_2|\cdots|S_k$. □

Example 3.11. Let $\sigma_2 = S_1|S_2 = s_0s_1s_2|s_0s_1$ as above. Then $\mathrm{Neg}(\sigma_2) = \{3, 2\}$ and

$$\sigma_2(3) = -1, \sigma_2(2) = -2 \quad , \quad \sigma_2(1) = 3, \sigma_2(4) = 4.$$

Lemma 3.12. *Let k be the number of steps in D_2. Then*

$$\mathrm{Des}(D_2) = \big\{ i < k : i \in \mathrm{Neg}(\sigma_2) \text{ and } i+1 \notin \mathrm{Neg}(\sigma_2) \big\}.$$

Proof. Let $S_1|S_2|\cdots|S_k$ be the expression for σ_2 as described above. Using Lemma 3.10, we get

$$\begin{aligned} i \in \mathrm{Des}(D_2) \quad &\Leftrightarrow \quad s_{i-1} \in S_j, s_i \notin S_j \text{ and } s_{i+1} \in S_{j-1} \text{ for some } j \\ &\Leftrightarrow \quad \sigma_2(i) \in \mathrm{Neg}(\sigma_2) \text{ and } i+1 \notin \mathrm{Neg}(\sigma_2). \end{aligned}$$

□

Lemma 3.13. *Let k be the number of steps in D_2. Then*

$$\mathrm{Des}(\sigma) = \mathrm{Des}(\sigma_1) \setminus \big\{ i < k : i \in \mathrm{Neg}(\sigma_2) \text{ and } i+1 \notin \mathrm{Neg}(\sigma_2) \big\}.$$

Proof. First, observe that $\mathrm{Neg}(\sigma_2) = \mathrm{Neg}(\sigma)$ and second, observe that the descents of σ and the descents of σ_1 which are larger than k coincide and that k is neither a descent of σ nor a descent of σ_1. By Lemma 3.4, we have to show that the descents of σ which are smaller than k are given by

$$\big\{ i < k : i \notin \mathrm{Neg}(\sigma) \text{ or } i+1 \in \mathrm{Neg}(\sigma) \big\}$$

and this can be deduced from Lemma 3.10. □

Lemma 3.14. $\mathrm{imaj}(\sigma) = \mathrm{imaj}(\sigma_1) + \mathrm{neg}(\sigma)$.

Proof. As $\sigma = \sigma_1\sigma_2$ and $\mathrm{iDes}(\sigma_2) = \emptyset$, we have

$$\mathrm{iDes}(\sigma) = \mathrm{iDes}(\sigma_1).$$

The lemma follows with the fact that $\mathrm{neg}(\sigma_1) = 0$. □

Proof of Eq. (8): Lemma 3.8 and Lemma 3.9 imply

$$\mathrm{maj}(D) = \mathrm{maj}(D_1) + \mathrm{maj}(D_2) - 2\,\mathrm{neg}(\sigma)$$

and by Lemma 3.12 and Lemma 3.13, we have

$$\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_1) - \mathrm{maj}(D_2) + \mathrm{neg}(\sigma).$$

Together with Lemma 3.14, Eq. (8) follows. □

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